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Waves generated by a source below a free surface in water of finite depth

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Abstract. The free-surface flow due to a submerged source in water of finite depth is considered. The fluid is assumed to be inviscid and incompressible. The problem is solved numerically by using a boundary integral equation formulation due to Hocking and Forbes [6]. The numerical results show that there is a train of waves on the free surface in accordance with the results of Mekias and Vanden-Broeck [5]. For small values of the Froude number, the amplitude of the waves is so small that the free surface is essentially flat in the far field. These waveless profiles agree with the calculations of Hocking and Forbes [6].

1. Introduction

In recent years, many solutions have been obtained for free-surface flows due to submerged sources (see for example Tuck and Vanden-Broeck [1], Hocking [2], Vanden-Broeck and Keller [3], Mekias and Vanden-Broeck ([4],[5]) and Hocking and Forbes [6]).

In this paper, we re-examine the flow due to a submerged source in a domain bounded above by a free surface and below by an horizontal bottom. We assume that there is a stagnation point on the free surface just above the source and that the flow is subcritical in the far field (see Fig. 1).

As we shall see the problem can be characterized by the Froude number

$$F_B = \left(\frac{m^2}{4gh_B^3}\right)^{1/2} \tag{1.1}$$

and the parameter

$$r = \frac{h_S}{h_B}, \tag{1.2}$$

where m is the strength of the source and g is the acceleration of gravity. The quantities h_B and h_S are the distances from the mean level of the free surface in the far field to the bottom and to the source, respectively.

The flow configuration of Fig. 1 was considered before by Hocking and Forbes [6] and by Mekias and Vanden-Broeck [5].

Hocking and Forbes [6] solved the problem numerically by a boundary-integral-equation method. For a given value of r, they obtain a solution for each value of F_B smaller than a limiting value F^L, at which the scheme fails to converge. The values of F_L depend on r and can be deduced from Fig. 6 in [6], where values of $F^L r^{-3/2}$ are plotted versus r. All the solutions of Hocking and Forbes [6] are characterized by a flat free surface in the far field.

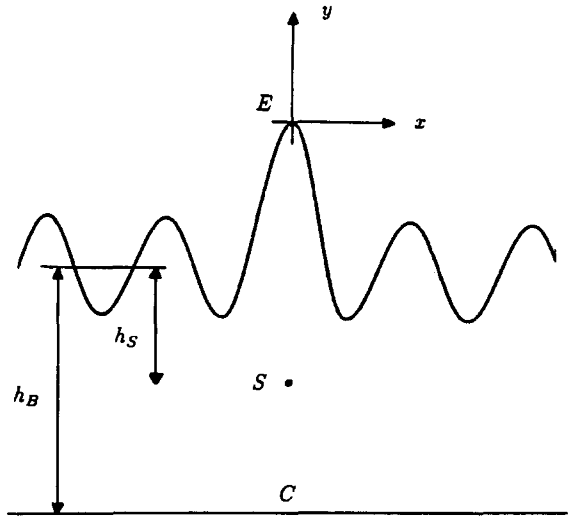

Figure 1. Sketch of the flow and of the coordinates.

Mekias and Vanden-Broeck [5] solved the problem numerically by using a different boundary-integral-equation formulation. They obtained solutions which are characterized by a train of waves in the far field (*i.e.* the free surface is not flat in the far field as in the computations of Hocking and Forbes [6]). For a given value of r, the solutions of Mekias and Vanden-Broeck [5] exist up to a critical value F^* of F_B at which the waves break. The values of F^* depend on r and can be deduced from Fig. 7 in [5], where the values of F^* are plotted versus the parameter $b = 1 - r$. For a given value of r, the value of F^* is larger than the value F^L.

The purpose of this paper is to reconcile the apparently contradictory results in [5] and [6]. We present new numerical results based on the boundary-integral-equation formulation of Hocking and Forbes [6]. These results confirm the findings of Mekias and Vanden-Broeck [5]. There is a train of waves in the far field. Our results also confirm the calculations of Hocking and Forbes [6] in the sense that, for $F_B < F^L$, the amplitude of the waves is so small that the free surface is essentially flat in the far field.

2. Formulation

We consider the fluid flow due to a source S of strength m. The fluid is inviscid and incompressible and the flow is irrotational. The flow domain is bounded above by a free surface and below by a horizontal bottom. We assume that there is a stagnation point at the point E just above the source. We introduce Cartesian coordinates with the origin at E (see Fig. 1). Gravity g is acting in the negative y direction. We define dimensionless variables by choosing $(m^2/8\pi^2 g)^{1/3}$ as the unit length and $(mg/\pi)^{1/3}$ as the unit velocity.

We introduce the complex potential function and the complex velocity

$$f = \phi + i\psi \tag{2.1}$$

$$w = u - iv. \tag{2.2}$$

Here u and v are the horizontal and vertical components of the velocity. Without loss of generality, we choose $\phi = 0$ at the stagnation point E and $\psi = 0$ on the free surface. In terms of the dimensionless variables, the dynamical condition on the free surface can be written as

$$u^2 + v^2 + y = 0. \tag{2.3}$$

We denote by $-H$ the ordinate of the bottom and we define the mean depth h_B by

$$H = h_B + \left(\frac{\pi}{h_B}\right)^2. \tag{2.4}$$

If the flow approaches a uniform stream in the far field, then h_B is the depth of the water in the far field. (this is the definition of h_B used in Hocking and Forbes [6] (see their Fig. 1)). This can easily be shown as follows. Let us denote by d, the uniform depth in the far field. Since the dimensionless flux is π, the velocity in the far field is then π/d. Substituting this value of the velocity in (2.3), we find that the difference of ordinates between the stagnation point and the level of the free surface at infinity is $(\pi/d)^2$. Therefore $H = d + (\pi/d)^2$. Comparing this with (2.4), we find that this implies $d = h_B$.

Using our dimensionless variables, we can now rewrite (1.1) as

$$F_B = \left(\frac{2\pi^2}{h_B^3}\right)^{1/2}. \tag{2.5}$$

Following Hocking and Forbes [6], we introduce the new variable

$$\zeta = e^f \tag{2.6}$$

and we define the function $\Omega(\zeta) = \delta(\zeta) + i\tau(\zeta)$ by

$$w(\zeta) = \frac{\pi}{h_B^*} e^{-i\Omega(\zeta)}. \tag{2.7}$$

When there are no waves on the free surface, then $h_B^* = h_B$. However, when there are waves, h_B^* is not equal to h_B and has to be found as part of the solution.

By using the Cauchy integral formula, Hocking and Forbes [6] derived the following relation between τ and δ on the free surface $1 < \zeta < \infty$

$$\tau(\zeta) = \frac{1}{2}\log\left[\frac{(1-\zeta)(\zeta_B - \zeta)}{\zeta^2}\right] + \frac{1}{\pi}\int_1^\infty \frac{\delta(\zeta_0)}{\zeta_0 - \zeta}\,d\zeta_0. \tag{2.8}$$

Here $\zeta_B = -e^{\phi_B}$, where ϕ_B is the value of the potential function at the point C with coordinates $(0, -H)$ (see Fig. 1). We obtain another relation between τ and δ on the free surface by substituting (2.7) in (2.3). This yields

$$\left(\frac{\pi}{h_B^*}\right)^2 e^{2\tau} + y = 0 \tag{2.9}$$

We obtain the values of x and y on the free surface by using (2.6) and the identity

$$\frac{d(x + iy)}{df} = w^{-1}. \tag{2.10}$$

This gives

$$x(\zeta) = \frac{h_B^*}{\pi} \int_1^{\zeta} \frac{e^{-\tau(\zeta_0)} \cos \delta(\zeta_0)}{\zeta_0} \, d\zeta_0, \tag{2.11}$$

$$y(\zeta) = \frac{h_B^*}{\pi} \int_1^{\zeta} \frac{e^{-\tau(\zeta_0)} \sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0. \tag{2.12}$$

Eqs. (2.8), (2.9) and (2.12) define a nonlinear integro-differential equation for the unknown function $\delta(\zeta)$ on the free surface $1 < \zeta < \infty$. This equation is essentially the same as the equation defined by the relations (2.7) and (2.9) in Hocking and Forbes [6]. The main difference is that we choose $y = 0$ at the stagnation point, whereas Hocking and Forbes choose $y = 0$ on the free surface at infinity.

3. Numerical procedure

Hocking and Forbes [6] presented a numerical procedure to solve the system of Eqs. (2.8), (2.9) and (2.12). They used equally spaced mesh points in the variable α defined by $\zeta = \sin^{-2} \alpha$. As they mention in their paper, this is not an appropriate choice if waves are present on the free surface, because there is then a singularity at $\alpha = 0$.

Here we use equally spaced points in the potential function ϕ. We first introduce the change of variables

$$\zeta = e^{\phi} \tag{3.1}$$

and rewrite (2.8) as

$$\tau'(\phi) = \frac{1}{2} \log \left[\frac{(1 - e^{\phi})(-e^{\phi_B} - e^{\phi})}{e^{2\phi}} \right] + \frac{1}{\pi} \int_0^{\infty} \frac{\delta'(\phi_0) e^{\phi_0}}{e^{\phi_0} - e^{\phi}} \, d\phi_0. \tag{3.2}$$

Similarly we rewrite (2.12) as

$$y'(\phi) = \frac{h_B^*}{\pi} \int_0^{\phi} e^{-\tau'(\phi_0)} \sin \delta'(\phi_0) \, d\phi_0 \tag{3.3}$$

Here $\tau'(\phi) = \tau(e^{\phi}), \delta'(\phi) = \delta(e^{\phi})$, etc. Next we introduce the mesh points

$$\phi_I = (I - 1)E, \quad I = 1, \ldots, N \tag{3.4}$$

and the corresponding unknowns

$$\delta_I = \delta'(\phi_I) \quad I = 1, \ldots, N \tag{3.5}$$

Since $\delta_1 = 0$, there are only $N - 1$ unknowns δ_I.

We evaluate the values τ_I^M of $\tau'(\phi)$ at the midpoints

$$\phi_I^M = \frac{\phi_I + \phi_{I+1}}{2}, \quad I = 1, \ldots, N - 1 \tag{3.6}$$

by applying the trapezoidal rule to the integral in (3.2) with a summation over the points ϕ_I. The symmetry of the quadrature and of the distribution of mesh points enabled us to evaluate the Cauchy principal value as if it were an ordinary integral.

Next we evaluate $y_I = y'(\phi_I)$ by applying again the trapezoidal rule to (3.3). This yields

$$y_1 = 0,$$

$$y_I = y_{I-1} + \frac{h_B^*}{\pi} e^{-\tau_I^M} \sin\left[\frac{\delta_I + \delta_{I-1}}{2}\right] E, \qquad I = 1, \ldots, N-1.$$

We evaluate the values of $\delta'(\phi)$ and $y'(\phi)$ at the midpoints (3.6) by the formulas

$$\delta_I^M = \frac{\delta_I + \delta_{I+1}}{2}, \qquad I = 1, \ldots, N-1, \tag{3.7}$$

$$y_I^M = \frac{y_I + y_{I+1}}{2}, \qquad I = 1, \ldots, N-1. \tag{3.8}$$

We now satisfy (2.9) at the midpoints ϕ_I^M, $I = 2, \ldots, N-1$ by substituting τ_I^M, (3.7) and (3.8) in (2.9). For given values of F_B (or h_B see (2.5)) and r, this yields $N-2$ nonlinear algebraic equations for the $N+1$ unknowns $\delta_2, \ldots, \delta_N, h_B^*$ and ϕ_B. One more equation is obtained by expressing $\delta_1 = 0$ in terms of δ_2 and δ_3 by a two-point extrapolation formula.

The last two equations are derived in the following way. First we calculate $\tau(\zeta)$ for $\zeta_B < \zeta < 1$ by integrating numerically (2.8) with the change of variable $\zeta_0 = e^{\phi_0}$. Since $|\delta| = \pi/2$ along the vertical line EC (see Fig. 1), (2.7) yields the values of w along EC. Substituting these values in the identity (2.10) and integrating, we obtain the values of $y(\zeta)$ along EC. In particular, we obtain the ordinates y_S and $-H$ of the source and the bottom. From Fig. 1, we see that

$$h_S = -y_S - H + h_B. \tag{3.9}$$

The last two equations are then obtained when we substitute the calculated values of H and h_S in (1.2) and (2.4).

The system of $N+1$ equations with $N+1$ unknowns is solved by Newton's method.

4. Discussion of the results

We used the numerical scheme described in Section 3 to compute solutions for various values of F_B and $r = 0.5$. We choose $r = 0.5$, because both Mekias and Vanden-Broeck [5] and Hocking and Forbes [6] present results for this value.

Most of the calculations were performed with $N = 510$ and $E = 0.0075$. We also calculated solutions with smaller values of E and larger values of N and checked that the results presented here are independent of E and N within graphical accuracy.

Typical free-surface profiles are shown in Fig. 2. Only half of the profiles are shown (the other half can be obtained by symmetry). There is a train of waves on the free surface in accordance with the results of Mekias and Vanden-Broeck [5]. However, for small values of F_B (e.g. $F_B = 0.15$), the waves are so small that they can hardly be seen on the figure and the profiles are flat in the far field as in the calculations of Hocking and Forbes [6].

Hocking and Forbes [6] presented a free-surface profile for $r = 0.5$ and $F_B = 0.15$ (see their Fig. 4). We found that their profile agrees with ours in Fig. 2. Similarly, Mekias and Vanden-Broeck [5] presented a profile for $r = 0.5$ and $F_B = 0.4$ (see Fig. 5 in [5] where $b = 1 - r$). Their profile coincides with ours in Fig. 2. The agreement of the results in [5] and [6] with ours constitute a check on the three schemes.

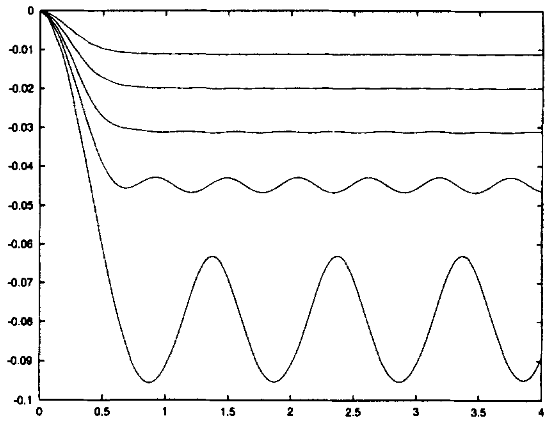

Figure 2. Computed free-surface profiles for $r = 0.5$. The values of the Froude number F_B from top to bottom are 0.15, 0.20, 0.25, 0.30 and 0.40.

All the free-surface profiles in [5] are flat in the far field (i.e. waveless). However, they are limited to values of $F_B < F^L$ (see introduction). For $F_B > F^L$, the scheme in [5] fails to converge. For $r = 0.5$, $F^L \approx 0.32$. Our numerical results in Fig. 2, show that solutions exist for $F_B > F^L$, but that they are characterized by waves of large amplitude in the far field. For $F_B < F^L$, the waves are small and can hardly be seen on the figures for $F_B < 0.25$. In fact, Mekias and Vanden-Broeck [6] showed that the amplitude of the waves is exponentially small as $F_B \to 0$.

Finally, let us mention that for each value of r, the solutions exist up to a value F^* of F_B at which the waves in the far field reach the Stokes limiting configuration with a 120° angle at their crests. Solutions for F_B close to F^* are presented in [5] and will not be duplicated here.

5. Conclusions

We have considered a free-surface flow due to a submerged source in water of finite depth. We have presented a numerical procedure based on the boundary integral equation of Hocking and Forbes [6]. The numerical results in [6] are limited to values of $F_B < F^L$. Our results extend these calculations to $F_B > F^L$. We have shown that there is a train of waves on the free surface in accordance with the findings of Mekias and Vanden-Broeck [5].

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